習題 5-1

1. $f(x)=e^x$
   (1) 定義域：R
   (2) 值域：$(0,\infty)$
   (3) 截距：$(0,1)$
   (4) 遞(　增　)函數
   (5) 當 $x\to\infty$，$a^x\to$(　∞　)
   (6) 當 $x\to 0^+$，$a^x\to$(　0　)
   (7) 連續函數？(　是　)
   (8) 一對一？(　是　)

2. $f(x)=\ln x$
   (1) 定義域：$(0,\infty)$
   (2) 值域：R
   (3) 截距：$(1,0)$
   (4) 遞(　減　)函數
   (5) 當 $x\to\infty$，$\log_a x\to$(　∞　)
   (6) 當 $x\to-\infty$，$\log_a x\to$(　$-\infty$　)
   (7) 連續函數？(　是　)
   (8) 一對一？(　是　)

3. 令 m,n 為正整數，x,y 為實數，則
   (1) $e^0=$(　1　)
   (2) $e^xe^y=$(　e^{x+y}　)
   (3) $\dfrac{e^x}{e^y}=$(　e^{x-y}　)
   (4) $(e^x)^y=$(　e^{xy}　)
   (5) $\dfrac{1}{e^x}=$(　e^{-x}　)
   (6) $e^{\frac{n}{m}}=$(　$\sqrt[m]{e^n}$　)

4. 令 x,y,r 為實數且 $x,y>0$，則
   (1) 乘法性質：$\ln xy=$(　$\ln x+\ln y$　)
   (2) 除法性質：$\ln\dfrac{x}{y}=$(　$\ln x-\ln y$　)
   (3) 指數性質：$\ln x^r=$(　$r\ln x$　)

5. (1) 近似值 $e \approx ($ 2.71828 $)$

(2) $\ln 1 = ($ 0 $)$

(3) $\ln e = ($ 1 $)$

(4) $\ln(e^x) = x, \quad \forall x \in R$

(5) $e^{(\ln x)} = x, \quad \forall x > 0$

(6) $\ln x = y \Rightarrow e^y = (x)$

6. 試繪下列函數的圖形

(1) $f(x) = 4^x$ 略。

(2) $g(x) = \log_4 x$ 略。

7. 求下列函數的定義域

(1) $f(x) = \ln(1-x)$

解 $D_f = \{x \mid 1-x > 0\} \Rightarrow D_f = \{x \mid x < 1\} \quad or \quad D_f = (-\infty, 1)$

(2) $g(x) = \log_2(x^2 - 1)$

解 $D_g = \{x \mid x^2 - 1 > 0\} \Rightarrow D_g = \{x \mid x > 1 \quad or \quad x < -1\} \quad or \quad D_f = (-\infty, -1) \cup (1, \infty)$

8. 求下列極限

(1) $\lim\limits_{n\to\infty}(1+\frac{2}{n})^n$

解 $\lim\limits_{n\to\infty}(1+\frac{2}{n})^n = \lim\limits_{n\to\infty}((1+\frac{1}{\frac{n}{2}})^{\frac{n}{2}})^2 \underbrace{=}_{m=\frac{n}{2}} \lim\limits_{m\to\infty}((1+\frac{1}{m})^m)^2 = (\underbrace{\lim\limits_{m\to\infty}(1+\frac{1}{m})^m}_{e})^2 = e^2$

(2) $\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}$

解 $\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}=\lim_{x\to\infty}\frac{e^{-x}(e^x-e^{-x})}{e^{-x}(e^x+e^{-x})}=\lim_{x\to\infty}\frac{1-e^{-2x}}{1+e^{-2x}}=1$

(3) $\lim_{x\to\infty}\frac{1-\ln x}{\ln x}$

解 $\lim_{x\to\infty}\frac{1-\ln x}{\ln x}=\lim_{x\to\infty}(\frac{1}{\ln x}-1)=-1$

習題 5-2

1. 填空：

(1) $\frac{d}{dx}\ln x = \frac{1}{(\quad x \quad)}$　　(2) $\frac{d}{dx}\ln u(x) = \frac{u'(x)}{(\quad u(x) \quad)}$

(3) $\frac{d}{dx}\log_a x = \frac{1}{x(\quad \ln a \quad)}$　　(4) $\frac{d}{dx}\log_a u(x) = \frac{(\quad u'(x) \quad)}{u(x)(\quad \ln a \quad)}$

2. 求下列導數

(1) $\frac{d}{dx}\ln(-x)$

解 $\frac{d}{dx}\ln(-x) = \frac{(-x)'}{-x} = \frac{-1}{-x} = \frac{1}{x}$

(2) $\frac{d}{dx}\ln\sqrt{x^3+x}$

解 $$\frac{d}{dx}\ln\sqrt{x^3+x} = \frac{d}{dx}\ln(x^3+x)^{\frac{1}{2}} = \frac{1}{2}\frac{d}{dx}\ln(x^3+x) = \frac{1}{2}\cdot\frac{\overbrace{(x^3+x)'}^{3x^2+1}}{x^3+x} = \frac{3x^2+1}{2(x^3+x)}$$

(3) $\frac{d}{dx}\ln(x(x+1)^5)$

解 $$\frac{d}{dx}\ln(x(x+1)^5) = \frac{d}{dx}(\ln x + 5\ln(x+1))$$
$$= \frac{d}{dx}\ln x + 5\frac{d}{dx}\ln(x+1) = \frac{1}{x} + \frac{5}{x+1} = \frac{6x+1}{x(x+1)}$$

(4) $\frac{d}{dx}\ln(\ln x)$

解 $\frac{d}{dx}\ln(\ln x)=\frac{\overbrace{(\ln x)'}^{\frac{1}{x}}}{\ln x}=\frac{1}{x\ln x}$

(5) $\frac{d}{dx}\ln\frac{x-1}{x}$

解 $\frac{d}{dx}\ln\frac{x-1}{x}=\frac{d}{dx}(\ln(x-1)-\ln x)=\frac{1}{x-1}-\frac{1}{x}=\frac{1}{x(x-1)}$

(6) $\frac{d}{dx}\ln\sqrt{\frac{x-1}{x}}$

解 $\frac{d}{dx}\ln\sqrt{\frac{x-1}{x}}=\frac{d}{dx}\frac{1}{2}(\ln(x-1)-\ln x)=\frac{1}{2x(x-1)}$

(7) $\frac{d}{dx}(\ln x)^2$

解 $\frac{d}{dx}(\ln x)^2=2\ln x\overbrace{(\ln x)'}^{\frac{1}{x}}=\frac{2\ln x}{x}$

(8) $\frac{d}{dx}\log_3 x$

解 $\frac{d}{dx}\log_3 x=\frac{1}{x\ln 3}$

(9) $\frac{d}{dx}\log(x+1)^5$

解 $\frac{d}{dx}\log(x+1)^5=\frac{d}{dx}5\log(x+1)=5\frac{(x+1)'}{(x+1)\ln 10}=\frac{5}{(x+1)\ln 10}$

(10) $\frac{d}{dx}x\log_2 x$

解 $\frac{d}{dx}(x\log_2 x)=\overbrace{(x)'}^{1}\log_2 x+x\overbrace{(\log_2 x)'}^{\frac{1}{x\ln 2}}=\log_2 x+\frac{1}{\ln 2}$

3. 試以隱函數微分求 dy/dx

(1) $x^2+\ln y+y^3=1$

解 $x^2+\ln y+y^3=1\Rightarrow\frac{d}{dx}(x^2+\ln y+y^3)=\frac{d}{dx}1$

$\Rightarrow 2x+\frac{1}{y}\frac{dy}{dx}+3y^2\frac{dy}{dx}=0\Rightarrow\frac{dy}{dx}=\frac{-2xy}{1+3y^3}$

(2) $y^2-\ln(xy)=x$

解 $y^2-\ln(xy)=x\Rightarrow\frac{d}{dx}(y^2-\ln(xy))=\frac{d}{dx}x$

$\Rightarrow 2y\frac{dy}{dx}-\frac{1}{xy}\underbrace{\frac{d}{dx}(xy)}_{y+x\frac{dy}{dx}}=1\Rightarrow\frac{dy}{dx}=\frac{xy+y}{2xy^2-x}$

習題 5-3

1. 填空：

(1) $\frac{d}{dx}e^{x^2}=2x(\quad e^{x^2}\quad)$　(2) $\frac{d}{dx}e^{u(x)}=e^{u(x)}(\quad u'(x)\quad)$

(3) $\frac{d}{dx}a^{x}=a^{x}(\quad \ln a\quad)$　(4) $\frac{d}{dx}a^{u(x)}=(\quad a^{u(x)}\quad)u'(x)\ln a$

2. 求下列導數

(1) $\frac{d}{dx}e^{-x^2}$

解 $\frac{d}{dx}e^{-x^2}=e^{-x^2}(-x^2)'=-2xe^{-x^2}$

(2) $\frac{d}{dx}e^{\frac{1}{x^2}}$

解 $\frac{d}{dx}e^{\frac{1}{x^2}}=e^{\frac{1}{x^2}}(\frac{1}{x^2})'=-2x^{-3}e^{\frac{1}{x^2}}$

(3) $\frac{d}{dx}x^2e^{2x}$

解 $\frac{d}{dx}x^2e^{2x}=\overbrace{(x^2)'}^{2x}e^{2x}+x^2\overbrace{(e^{2x})'}^{2e^{2x}}=2xe^{2x}+2x^2e^{2x}=2x(1+x)e^{2x}$

(4) $\frac{d}{dx}(x+1)e^{-x}$

解 $\frac{d}{dx}(x+1)e^{-x}=\overbrace{(x+1)'}^{1}e^{-x}+(x+1)\overbrace{(e^{-x})'}^{-e^{-x}}=e^{-x}-(x+1)e^{-x}=-xe^{-x}$

(5) $\frac{d}{dx}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

解 $\frac{d}{dx}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\overbrace{(-\frac{x^2}{2})'}^{-x}=\frac{-x}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

(6) $\frac{d}{dx}5^x$

解 $\frac{d}{dx}5^x=5^x\ln 5$

(7) $\frac{d}{dx}x2^x$

解 $\frac{d}{dx}x2^x=\overbrace{(x)'}^{1}2^x+x\overbrace{(2^x)'}^{2^x\ln 2}=2^x+x(\ln 2)2^x=(1+x\ln 2)2^x$

(8) $\frac{d}{dx}3^{2x-1}$

解 $\frac{d}{dx}3^{2x-1}=3^{2x-1}\overbrace{(2x-1)'}^{2}\ln 3=2(\ln 3)3^{2x-1}$

(9) $\frac{d}{dx}(e^x\ln x)$

解 $\frac{d}{dx}(e^x\ln x)=\overbrace{(e^x)'}^{e^x}\ln x+e^x\overbrace{(\ln x)'}^{\frac{1}{x}}=e^x\ln x+\frac{e^x}{x}=(\frac{1}{x}+\ln x)e^x$

(10) $\frac{d}{dx}e^{\ln x^2}$

解 $\frac{d}{dx}e^{\ln x^2}=e^{\ln x^2}\overbrace{(\ln x^2)'}^{\frac{2}{x}}=\frac{2e^{\ln x^2}}{x}$

3. 求下列導數

(1) $y=(\ln x)^x$

解 $\dfrac{dy}{dx}=\dfrac{d}{dx}(\ln x)^x=\dfrac{d}{dx}e^{\ln(\ln x)^x}=\dfrac{d}{dx}e^{x\ln(\ln x)}$

$$=\overbrace{e^{x\ln(\ln x)}}^{(\ln x)^x}\cdot\underbrace{(x\ln(\ln x))'}_{\underbrace{(x)'}_{1}\ln(\ln x)+x\underbrace{(\ln(\ln x))'}_{\frac{1}{x\ln x}}}=(\ln x)^x(\ln(\ln x)+\frac{1}{\ln x})$$

(2) $x^2-y^2=2^x$

解 $\dfrac{d}{dx}(x^2-y^2)=\dfrac{d}{dx}2^x\Rightarrow 2x-2y\dfrac{dy}{dx}=2^x\ln 2\Rightarrow\dfrac{dy}{dx}=\dfrac{2x-2^x\ln 2}{2y}$